Unary algebras and Constraint Satisfaction Problem

Przemysław Broniek
Algorithmics Research Group
Jagiellonian University
broniek@tcs.uj.edu.pl

Consider a finite algebra $\mathbf{A} = (A, F)$. A term over $\mathbf{A}$ is any proper expression built with variables and operations from F. Then the SysTermSat($\mathbf{A}$) is a problem in which the instance consists of a finite set of term equations $S = \{t_i(x_1, \ldots, x_n) \approx s_i(x_1, \ldots, x_n)\}_i$ over the algebra $\mathbf{A}$. The question is whether the system S of equations has a solution, that is a valuation of variables satisfying all equations.

A finite relational structure $\mathbf{D}$ is a pair $(D, \mathcal{R})$, where $\mathcal{R}$ is a finite family of relations over a domain set D. The Constraint Satisfaction Problem over $\mathbf{D}$, denoted CSP($\mathbf{D}$), is a problem in which the instance consists of:

- a finite set of variables V,
- a finite set of constraints $\mathcal{C} = \{C_i\}_i$, each of the form:

$$C_i = (R_i, (X^i_1, \ldots, X^i_{n_i})),$$

where $R_i \in \mathcal{R}$ and n_i is the arity of R_i. Each of X^i_j is chosen from V. The question is whether there exists a solution, that is a function $s : V \to A$ such that $(s(X^i_1), \ldots, s(X^i_{n_i})) \in R_i$ for all constraints $C_i \in \mathcal{C}$.

The computational complexity of the above problems has been studied for many classes of algebras (or relational structures) and is still open in general. We restrict ourselves to unary algebras (with only unary operations) and show that for every relational structure $\mathbf{D}$ the problem CSP($\mathbf{D}$) is polynomially equivalent to SysTermSat($\widetilde{\mathbf{D}}$), where $\widetilde{\mathbf{D}}$ is a unary algebra constructed from $\mathbf{D}$. We also present theorem reducing SysTermSat of arbitrary large unary algebras, which term operations take only few values, to the Constraint Satisfaction Problem over domains with these few values.