DPLL - style Rules for Deciding Generalized Satisfiability

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Many existence problems in mathematics reduce to **SAT** — the problem of determining satisfiability of a family (not necessarily finite) of propositional formulae. One of these is – via Herbrand Theorem – also a problem of provability in the first order calculus.

Since every propositional formula is equivalent to a Conjunctive Normal Form, the general problem of satisfiability is usually reduced to satisfiability of families of clauses – sets of literals: variables and their negations.

Martin Davis and Hilary Putnam ([4]) in 1960 introduced a procedure for determining satisfiability of propositional formulas in conjunctive normal form which was designed to be run on computers. Shortly thereafter Davis, Longemann and Loveland ([3]) implemented a modified version of this algorithm. This procedure, known as **DPLL**, is still considered to be the most successful procedure for satisfiablity testing and is the basis for most efficient **SAT**-solvers.

Cowen in [1] and Kolany in [5] considered a generalization of satisfiability to the case of hypergraphs and families of sets of their vertices (generalized clauses). Then, Cowen noticed some similarity between hypergraph satisfiability and Schrijver's existence of transversals (see [6]) and introduced in [2] the following property **S** for a bihypergraph $\langle V, \mathcal{E}, \mathcal{F}\rangle$:

Definition 1 *We say that $\mathcal{H} = \langle V, \mathcal{E}, \mathcal{F}\rangle$ has property* **S** *[or that $\mathcal{H}$ is satisfiable] iff there exists a set $X \subseteq V$ satisfying $f \cap X \neq \emptyset$, for $f \in \mathcal{F}$, and $e \not\subseteq X$, for $e \in \mathcal{E}$. We also then say that X satisfies $\mathcal{H}$.*

We easily notice that satisfiability of $\langle V, \mathcal{E}, \mathcal{F}\rangle$ is equivalent to satisfiability of $\langle \bigcup\mathcal{E} \cup \bigcup\mathcal{F}, \mathcal{E}, \mathcal{F}\rangle$ which means that satisfiability is a property of the pair $\langle\mathcal{E}, \mathcal{F}\rangle$ rather then of the triple $\langle V, \mathcal{E}, \mathcal{F}\rangle$. This also legitimitizes using the notation $\mathbf{S}(\mathcal{E}\,;\mathcal{F})$ to denote having property S by the apropriate bihypergraph. At first, let us notice that if X satisfies $\langle V, \mathcal{E}, \mathcal{F}\rangle$ then $\mathcal{V} \setminus X$ satisfies $\langle V, \mathcal{F}, \mathcal{E}\rangle$ which gives the following duality principle

Theorem 1

$$\mathbf{S}(\mathcal{E}\,;\mathcal{F}) \iff \mathbf{S}(\mathcal{F}\,;\mathcal{E})$$

As a matter of fact **DPLL** procedure uses repeatedly the three following rules:

1. **Unit prune rule**

   If there exists a singleton clause $\{\ell\}$ then delete all clauses containing ℓ from the considered set of clauses and remove its opposite $\ell^\star$ from the others. This does not affect the satisfiability.

2. **Pure Literal Rule**

   If there exists a literal ℓ for which $\ell^\star$ does not occur in the set of clauses, delete all the clauses which contain ℓ. This does not affect the satisfiability.

and

3. **Splitting Rule**

   Choose a literal ℓ and consider two sets of clauses. The first is obtained by deleting all clauses containing ℓ and removing $\ell^\star$ from the other clauses and teh second is obtained by deleting all the clauses containing $\ell^\star$ and removing ℓ from the others. Then the family is satisfiable if and only if one of them is satisfiable.

Analyzing the above rules gave rise to the following ones for property S. Their names are taken from their propositional logic counterparts:

Theorem 2

1. (Unit Prune Rule, **UPR**) *Let $v \in V$. Then:*

$$\mathbf{S}(\mathcal{E}\,;\mathcal{F},\underline{v}) \iff \mathbf{S}(\mathcal{E}\setminus\!\!\setminus\underline{v}\,;\mathcal{F}^{\underline{v}})$$

   where $\mathcal{F},\underline{v} = \mathcal{F}\cup\{v\}$; $\mathcal{E}\setminus\!\!\setminus\underline{v} = \{e\setminus\{v\} : e \in \mathcal{E}\}$ and $\mathcal{F}^{\underline{v}} = \{f \in \mathcal{F} : v \notin f\}$.

2. (Pure Literal Rule, **PLR**)

$$\mathbf{S}(\mathcal{E}\,;\mathcal{F}) \iff \mathbf{S}(\mathcal{E}_{[\mathcal{F}]}\,;\mathcal{F}_{[\mathcal{E}_{[\mathcal{F}]}]})$$

   where $\mathcal{A}_{[\mathcal{B}]} = \{\alpha \in \mathcal{A} :\ \alpha \subseteq \bigcup\mathcal{B}\}$ for $\mathcal{A}, \mathcal{B} \subseteq \wp(V)$,

3. (Splitting Rule, **SPL**) *Let $f \subseteq V$ and let $f = \alpha\cup\beta$. Then:*

$$\mathbf{S}(\mathcal{E}\,;\mathcal{F}, f) \iff \mathbf{S}(\mathcal{E}\,;\mathcal{F},\alpha) \vee \mathbf{S}(\mathcal{E}\,;\mathcal{F},\beta)$$

   where $\mathcal{F}, f = \mathcal{F}\cup\{f\}$; $\mathcal{F},\alpha = \mathcal{F}\cup\{\alpha\}$ and $\mathcal{F},\beta = \mathcal{F}\cup\{\beta\}$.

Pure Literal Rule can be inferred from two simpler principles:

Theorem 3

1. *Let* $X \cap \bigcup \mathcal{F} = \emptyset$ *and let* $\mathcal{E}^X = \{e \in \mathcal{E} : X \cap e = \emptyset\}$*. Then* $\mathbf{S}(\,\mathcal{E}\,;\,\mathcal{F}\,)$ *iff* $\mathbf{S}(\,\mathcal{E}^X\,;\,\mathcal{F}\,)$*.*
2. *Let* $\mathcal{A} \subseteq \wp(V)$*. If* $\mathcal{E} \subseteq \mathcal{A}$*, then* $\mathbf{S}(\,\mathcal{E}\,;\,\mathcal{F}\,)$ *iff* $\mathbf{S}(\,\mathcal{E}\,;\,\mathcal{F}_{[\mathcal{A}]}\,)$*.*

The Splitting Rule yields the following recursive relations:

Theorem 4

1. *Let* $v \in f \subseteq V$*. Then*

$$\mathbf{S}(\,\mathcal{E}\,;\,\mathcal{F}, f\,) \iff \mathbf{S}(\,\mathcal{E}\backslash\!\!\backslash\,\underline{v}\,;\,\mathcal{F}^{\underline{v}}\,) \;\vee\; \mathbf{S}(\,\mathcal{E}\,;\,\mathcal{F}, f \setminus \underline{v}\,)$$

2. *Let* $v \in V$ *and let* $\mathcal{E}_{\underline{v}}$ *or* $\mathcal{F}_{\underline{v}}$ *be finite. Then*

$$\mathbf{S}(\,\mathcal{E}\,;\,\mathcal{F}\,) \iff \mathbf{S}(\,\mathcal{E}^{\underline{v}}\,;\,\mathcal{F}\backslash\!\!\backslash\,\underline{v}\,) \;\vee\; \mathbf{S}(\,\mathcal{E}\backslash\!\!\backslash\,\underline{v}\,;\,\mathcal{F}^{\underline{v}}\,)$$

Also the following Superset Rule is considered,

Theorem 5 (Superset Rule, SUP) *If* $e \subseteq f$*,* $e \neq f$*, then* $\mathbf{S}(\,\mathcal{E}, e, f\,;\,\mathcal{F}\,)$ *iff* $\mathbf{S}(\,\mathcal{E}, e\,;\,\mathcal{F}\,)$

Example 1 *To check colorability of the graph* $G = \langle V, E\rangle$*, where* $V = \{a, b, c, d, e\}$ *and*

$$E = \{\{a, e\}, \{a, d\}, \{b, c\}, \{b, e\}, \{c, d\}, \{d, e\}\},$$

we must check satisfiability of $\langle\, V\,,\, E\,,\, E\,\rangle$ *(see [5]). The following table proves it is not satisfiable (here we write* $\mathcal{E} \,\|\, \mathcal{F}$ *, instead of* $\mathbf{S}(\,\mathcal{E}\,;\,\mathcal{F}\,)$*, for readibility):*

<table>
<tr><td colspan="2">$ae, ad, bc, be, cd, de \,\|\, ae, ad, bc, be, cd, de$ $\mathbf{SPL}^{a,d}$</td></tr>
<tr><td>$ae, a, bc, be, cd, de \,\|\, ae, ad, bc, be, cd, de$ $\mathbf{UPR}(a)$</td><td>$ae, d, bc, be, cd, de \,\|\, ae, ad, bc, be, cd, de$ $\mathbf{UPR}(d)$</td></tr>
<tr><td>$bc, be, cd, de \,\|\, e, bc, be, cd, de$ $\mathbf{UPR}(e, d)$</td><td>$ae, d, bc, be \,\|\, ae, a, bc, be, c, e$ $\mathbf{UPR}(a, c, e)$</td></tr>
<tr><td>$a, bc, b, c, \{\} \,\|\, bc$</td><td>$\{\}, d, b, b \,\|\, -$</td></tr>
</table>

Since in both branches there are empty clauses to be satisfied, the original graph is not colorable.

Example 2 *To check* $(2, 1)$*-colorability of the previous graph, one needs to decide colorability of the hypergraph* $\langle V, \mathcal{E}\rangle$*, where* $V = \{a, b, c, d, e\}$ *and* $\mathcal{E} = \{\{a, d, e\}, \{b, c, e\}, \{b, c, d\}, \{c, d, e\}, \{b, d, e\}, \{a, b, e\}\}$ *(see [5]). Thus, we need to check satisfiability of* $\langle\, V\,,\, \mathcal{E}\,,\, \mathcal{E}\,\rangle$*. We have:*

<table>
<tr><td colspan="5">$ade, bce, bcd, cde, bde, abe \,\|\, ade, bce, bcd, cde, bde, abe$ $\mathbf{SR}^{ad,e}$</td></tr>
<tr><td colspan="4">$e, bce, bcd, cde, bde, abe \,\|\, ade, bce, bcd, cde, bde, abe$ $\mathbf{UPR}(e)$
$bcd \,\|\, ad, bc, bcd, cd, bd, ab$ $\mathbf{SUP}(bc)$
$bcd \,\|\, ad, bc, cd, bd, ab$ $\mathbf{PLR}(ad, ab)$
$bcd \,\|\, bc, cd, bd$ $\mathbf{SR}_{c,d}$</td><td rowspan="4">$ad, bce, bcd, cde, bde, abe \,\|\, ade, bce, bcd, cde, bde, abe$ $\mathbf{SR}^{b,ce}$
...</td></tr>
<tr><td colspan="2">$bcd \,\|\, bc, c, bd$ $\mathbf{UPR}(c)$
$bd \,\|\, bd$ $\mathbf{SR}^{b,d}$</td><td colspan="2" rowspan="2">$bcd, \,\|\, bc, d, bd$ $\mathbf{UPR}(d)$
$bc \,\|\, bc$ $\mathbf{SR}^{b,c}$

...</td></tr>
<tr><td>$b \,\|\, bd$ $\mathbf{UPR}(b)$
$- \;\|\, d$</td><td>$d \,\|\, bd$ $\mathbf{UPR}(d)$
$- \;\|\, b$</td></tr>
</table>

Since, for instance, $\langle\, V\,,\, \emptyset\,,\, \{\{d\}\}\,\rangle$ *is satisfiable, the original graph is* $(2, 1)$*-colorable.*

References

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